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EXACT SOLUTION OF A PARTICULAR PROBLEM OF OSCILLATIONS OF A SYSTEM WITH RANDOM PARAMETRIC EXCITATION*

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A second order system with external and parametric perturbations of the white noise type is considered. An exact analytic solution of the steady Fokker—Plank—Kolmogorov equation is obtained for this system at some special relation between the coefficients of modulation intensity with respect to the coordinate and velocity. The solution determines the power relationship for the combined probability density of the coordinate and velocity.

The obtained solution can be used for checking various approximate methods of investigation of problems of dynamics of systems with random varying parameters. Note that the majority of these methods are local based on the assumption of closeness (in one sense or another) of the motion of a given system to that of a system with constant parameters (see, e.g., /1/ and the cited there bibliography). For some systems it is possible to effect a nonlocal analysis using the method of Markovian processes in conjunction with the method of averaging. Such analysis was carried out in /2/ for a second order system based on the solution of the Fokker—Plank—Kolmogorov equation for the steady one-dimensional probability density of the amplitude.

Let us consider the second order system

$$x^{\cdot\cdot} + 2\alpha x^{\cdot}[1 + \eta(t)] + \Omega^2 x[1 + \xi(t)] = \zeta(t) \tag{1}$$

where $\xi(t)$, $\eta(t)$, $\zeta(t)$ are statistically independent steady Gaussian centralized random processes of the white noise type in the sense of Stratonovich /3,4/ with the respective intensity coefficients D_ξ, D_η, D_ζ. We rewrite (1) in the form of the equivalent system of two Ito's stochastic equations

$$dx = y\,dt,\ dy = [-(2\alpha - 2\alpha^2 D_\eta)\,y - \Omega^2 x]\,dt - \Omega^2 x D_\xi^{1/2} d\Xi - 2\alpha y D_\eta^{1/2} d\mathrm{H} + D_\zeta^{1/2}\,dZ$$

where Ξ, H, Z are independent Wiener's processes, and compose the steady Fokker—Plank—Kolmogorov equation for the combined probability density $p(x, y)$ of processes $x(t)$ and $y(t)$

$$y\frac{\partial p}{\partial x} = \Omega^2 x\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}[(2\alpha - 2\alpha^2 D_\eta)\,yp] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[(4\alpha^2 y^2 D_\eta + \Omega^4 x^2 D_\xi + D_\zeta)\,p] \tag{2}$$

A direct substitution will show that when coefficients D_ξ, D_η are linked by the relation

$$\Omega^2 D_\xi = 4\alpha^2 D_\eta \tag{3}$$

then Eq.(2) has the following solution:

$$p(x, y) = C(\varkappa + x^2 + y^2/\Omega^2)^{-\delta} \tag{4}$$

$$\varkappa = D_\zeta/(D_\xi\Omega^4),\quad \delta = 2\alpha/(D_\xi\Omega^2) + 1/2 \tag{5}$$

which is valid over the whole x, y -plane and is damped at infinity.

Function $p(x, y)$ defined by formula (4) actually represents the combined probability density of the coordinate and velocity, when the normalization condition $\delta > 1$ is satisfied. The normalization constant C is then

$$C = \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y)\,dx\,dy\right]^{-1} = (\pi\Omega)^{-1}(\delta - 1)\,\varkappa^{\delta-1} \tag{6}$$

The indicated condition $\delta > 1$ of existence of steady probability density $p(x, y)$ must coincide with the stability condition for system (1) with respect to probability. This fairly evident statement can be strictly proved, at least in the case of small α, D_ξ, D_η, by passing with the use of the asymptotic method of averaging to the first order equation for the amplitude of process $x(t)$, and applying the method of stability analysis described in /4/.

Moreover, when $\varkappa = 0$, the probability density (4) has (in the case of $\delta > 1$) a nonintegrable singularity at point $x = 0, y = 0$, i.e. $p(x, y)$ degenerates into a delta function $\delta(0, 0)$.

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Physically this corresponds to total absence of oscillations of the stochastically stable (when $\delta > 1$) system (1) in the absence of external excitation. Note, also, that at the limit as $D_\xi \to 0$ we obtain, as expected, from (4) and (5) the expression for the normal probability density $p(x, y)$.

By passing in (4) to new variables A and φ using formulas $x = A\cos\varphi$, $y = \Omega A \sin\varphi$ it is possible to determine in conformity with the rules of determination of probability density of functions of random quantities /5/, the combined probability density $p(A, \varphi)$. Integrating the latter with respect to φ from zero to 2π we obtain the one-dimensional steady probability density of the amplitude

$$p(A) = \frac{2A(\delta - 1)}{\varkappa(1 + A^2/\varkappa)^\delta} \tag{7}$$

The same amplitude distribution (7) is obtained generally for small $\alpha, D_\xi, D_\eta, D_\zeta$ (when equality (3) is not satisfied) using the asymptotic method of averaging; in /2/ distribution (7) was obtained for $D_\eta = 0$. Generally the dependence of parameters $\varkappa, \delta$ of that distribution on parameters $\alpha, \Omega, D_\xi, D_\eta, D_\zeta$ is, obviously, different from that determined by (5). When equality (3) is satisfied, the approximate distribution $p(A)$ determined by the method of averaging coincides with the exact one.

Integration of (4) with respect to y yields the one-dimensional steady probability distribution of coordinate $x(t)$

$$p(x) = \int_{-\infty}^{\infty} p(x, y)\, dy = \frac{\varkappa^{\delta-1}\Gamma(\delta - 1/2)}{\sqrt{\pi}\,\Gamma(\delta - 1)} (\varkappa + x^2)^{-(\delta - 1/2)} \tag{8}$$

where Γ is the gamma function. Let us determine the moments of power distribution (8). All moments of odd order turn out to be zero, while for moments of order $2k$ we obtain

$$\langle x^{2k} \rangle = \int_{-\infty}^{\infty} x^{2k}\, p(x)\, dx = \frac{\varkappa^k \Gamma(k + 1/2)\,\Gamma(\delta - k - 1)}{\sqrt{\pi}\,\Gamma(\delta - 1)} \tag{9}$$

where the angle brackets denote everywhere averaging. It will be seen that moments of order $2k$ of the process $x(t)$ are finite only when $\delta > k + 1$. The inequality $\delta > k + 1$ is the condition of stochastic stability of system (1) with respect to moments of order $2k$. When $1 < \delta < 2$ process $x(t)$ is unstable in the quadratic mean, although steady probability density of this process does exist. Note that such steady probability densities were observed in numerical simulation of system (1) with $D_\eta = 0$ on a computer /2/. In the light of the above it is not surprising that in the case of $1 < \delta < 2$ it was not possible to obtain consistent estimates of dispersion for process $x(t)$; the estimates considerably varied from one segment of the process realization $x(t)$ to another.

On the basis of formula (4) it is also possible to determine the mixed moments of the coordinate and velocity. Distribution (4) is an interesting example of a situation in which a steady random process and its first derivative are not statistically independent. Indeed, although $\langle xy \rangle = 0$ (processes $x(t)$ and $y(t)$ are not correlated), the fourth order mixed moment

$$\langle x^2 y^2 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 y^2 p(x, y)\, dx\, dy = \frac{\varkappa^2\Omega^2}{4(\delta - 2)(\delta - 3)}$$

coincides with the product $\langle x^2\rangle\langle y^2\rangle = (\langle x^2\rangle)^2\Omega^2 = 1/_4\varkappa^2\Omega^2(\delta - 2)^{-2}$ only at the limit as $\delta \to \infty$ which corresponds to the asymptotically normal distribution.

In concluding, let us determine, using the formula in /5/ the average number of intersections $n(x_0)$ of process $x(t)$ with the level $x = x_0$ with positive derivative. We have

$$n(x_0) = \int_0^{\infty} y p(x_0, y)\, dy = (\Omega/2\pi)(1 + x_0^2/\varkappa)^{-(\delta-1)} \tag{10}$$

At the limit as $D_\xi \to 0$ we obtain, as expected

$$n(x_0) = (\Omega/2\pi)\exp(-2\alpha\Omega^2 x_0^2/D_\zeta)$$

Formula (10) clearly shows that with the approach to the boundary $\delta = 1$ of stochastic stability of system (1) a continuous increase of the level of parametric oscillation intensification of oscillations induced by external random perturbations $\zeta(t)$, takes place.

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